Name: ______________________________

Complex Analysis
Candidacy Exam, Spring 2005

Choose 3 problems from 1-6 and 3 problems from 6-12.

1. Define analytic functions. What is the Cauchy-Riemann condition? Show that if a function is analytic and $f = u + iv$, then u and v are harmonic.

2. Show that if f is analytic in Ω and $|f|$=constant, then f is a constant.

3. State and prove the Morera Theorem.

4. Let $D = \{|z| < 1\}$ and f be an analytic function in D. We assume that $|f(z)| \leq 1$ in D. State and prove the Schwarz Lemma. Prove that

$$\frac{|f'(z)|}{(1 - |f(z)|^2)} \leq \frac{1}{1 - |z|^2}.$$

5. Evaluate the integral (Choose one)

(1)
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx.$$

(2)
$$\int_0^{2\pi} \frac{d\theta}{z + \cos\theta}.$$

6. Suppose $f(z)$ is analytic in a domain Ω in $\mathbb{C}$ and γ is a closed curve in Ω. We assume that $|f(z) - 1| < 1$ for every $z \in \gamma$. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

7. a). What is a simply connected domain? State the Riemann Mapping Theorem.

b). Let Ω be the half disk defined by $\Omega = \{z = x + iy \mid |z| < 1, x > 0\}$. Find a conformal map that will map Ω unto the unit disk.

8. a). Define the principal branch of $\log z$ for $\mathbb{C} \setminus [-\infty, 0]$.
b). Can one define a single-valued branch of $\log z$ in the domain $\Omega = \mathbb{C} \setminus [-1, 1]$?
c). Can one define a single-valued analytic branch of $\sqrt{1 - z^2}$ in Ω?

9. Let $f_n(z)$ be a sequence of analytic functions in a domain Ω in $\mathbb{C}$. Show that if f_n is uniformly bounded on Ω, then there exists a convergent subsequence which converges uniformly on compact subsets in Ω to some function f. Show that f is analytic in Ω.

10. Let $f_n(z)$ be a sequence of analytic functions in a domain Ω in $\mathbb{C}$. If f_n converges to a nonconstant function f in Ω. Show that if f_n is one to one in Ω, then f is one to one in Ω.

11. Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ is a polynomial.

12. a). Find all one-to-one linear mappings of the unit disc onto itself.

b). Use the Schwarz lemma to show that every one-to-one conformal mappping of the unit disc onto itself is given by a linear map.